MAA K08

1\. a) $2x^2 = x + 1$

$2x^2 - x - 1 = 0$

$x = \frac{1 \pm \sqrt{1 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} = \frac{1 \pm 3}{4}$ $\quad \underline{\underline{x = 1 \vee x = -\frac{1}{2}}}$

b) $\frac{x}{6} - \frac{x-2}{3} = \frac{5}{12} \quad | \cdot 12$

$2x - 4(x-2) = 5$

$2x - 4x + 8 = 5$

$-2x = -3$

$\underline{\underline{x = 1\frac{1}{2}}}$

c) $x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}, \quad t \in \mathbb{R}$

$x^2 + y^2 = \frac{1 - 2t^2 + t^4 + 4t^2}{(1+t^2)^2} = \frac{1 + 2t^2 + t^4}{(1+t^2)^2} = \frac{(1+t^2)^2}{(1+t^2)^2} = \underline{\underline{1}}$

2\. a) $\begin{cases} 2x + y = 8 & | \cdot(-2) \\ 3x + 2y = 5 \end{cases}$

$\begin{cases} -4x - 2y = -16 \\ 3x + 2y = 5 \end{cases}$

$-x = -11$

$x = 11$

$2 \cdot 11 + y = 8$

$y = 8 - 22$

$y = -14$ $\quad \underline{\underline{V: (11, -14)}}$

b) $5^{5x-5} = 125$

$5^{5x-5} = 5^3$

$5x - 5 = 3$

$5x = 8$

$\underline{\underline{x = 1\frac{3}{5}}}$

c) $|3x - 2| = 5$

$3x - 2 = \pm 5$

$3x = 7 \quad \vee \quad 3x = -3$

$\underline{x = 2\frac{1}{3} \quad \vee \quad x = -1}$

MAA K08

3\. a) $f(x) = x^{-4}$

$f'(x) = -4x^{-5} = -\frac{4}{x^5}$ $\quad \int x^{-4}\,dx = -\frac{1}{3}x^{-3} + C = -\frac{1}{3x^3} + C$

$f(x) = x^{-1}$

$f'(x) = -x^{-2}$ $\quad \int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$

$f(x) = x^2$

$f'(x) = 2x$ $\quad \int x^2\,dx = \frac{1}{3}x^3 + C$

b) $f(x) = \frac{2 + \sin x}{2 + \cos x}$

$f'(x) = \frac{\cos x(2 + \cos x) - (2 + \sin x)(-\sin x)}{(2 + \cos x)^2}$

$= \frac{2\cos x + \cos^2 x + 2\sin x + \sin^2 x}{(2 + \cos x)^2}$

$= \frac{2\sin x + 2\cos x + 1}{(2 + \cos x)^2}$

$f'(\frac{\pi}{2}) = \frac{2 \cdot 1 + 2 \cdot 0 + 1}{(2 + 0)^2} = \underline{\underline{\frac{3}{4}}}$

4\.

veroton hinta	a	a
verollinen "	$1{,}22a$	$1{,}08a$

$\frac{1{,}08a}{1{,}22a} = 0{,}88524\ldots$

$1 - 0{,}88524\ldots = 0{,}1147\ldots \approx \underline{\underline{11{,}5\,\%}}$

5\. Kaikki järjestykset $5!$

$P(\text{Oikea järjestys}) = \frac{1}{5!} = \frac{1}{120}$ $\quad P(\text{Väärä järj.}) = \frac{119}{120}$

$P(\text{ainakin kerran o}) = 1 - P(\text{ei kertaakaan})$

$= 1 - (\frac{119}{120})^7 = 1 - 0{,}9431\ldots$

$= 0{,}0568\ldots$

$\underline{\underline{V: 5{,}7\,\%}}$

MAA K08

6. $\bar{a} = 5\bar{i} - 2\bar{j} \qquad \bar{b} = 3\bar{i} + t\bar{j}$

$\bar{a} \parallel \bar{b} \Leftrightarrow 5\bar{i} - 2\bar{j} = s(3\bar{i} + t\bar{j})$

$5\bar{i} - 2\bar{j} = 3s\bar{i} + st\bar{j}$

$\begin{cases} 5 = 3s \rightarrow s = \frac{5}{3} \\ -2 = st \end{cases} \Rightarrow \frac{5}{3}t = -2 \Leftrightarrow \underline{\underline{t = -1\frac{1}{5}}}$

tai kulmakertoimet: $-\frac{2}{5} = \frac{t}{3} \Rightarrow \underline{\underline{t = -1\frac{1}{5}}}$

$\bar{a} \perp \bar{b} \Leftrightarrow \bar{a} \cdot \bar{b} = 0$

$5 \cdot 3 + (-2) \cdot t = 0$

$-2t = -15$

$\underline{\underline{t = 7\frac{1}{2}}}$

7. a) $\begin{cases} y = x^2 - 3 \\ y = -x^2 + 2x + 1 \end{cases}$

$x^2 - 3 = -x^2 + 2x + 1$

$2x^2 - 2x - 4 = 0 \quad | :2$

$x^2 - x - 2 = 0$

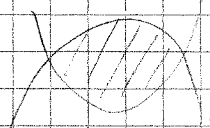
$x = \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-2)}}{2 \cdot 1} = \frac{1 \pm 3}{2} \begin{cases} 2 \\ -1 \end{cases}$

$y(2) = 2^2 - 3 = 1$

$y(-1) = (-1)^2 - 3 = -2$

$\underline{\underline{V: (2, 1) \text{ ja } (-1, -2)}}$

b)



$y = -x^2 + 2x + 1$

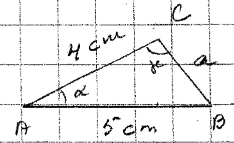
$A = \int_{-1}^{2} (-x^2 + 2x + 1 - (x^2 - 3))\,dx = \int_{-1}^{2} (-2x^2 + 2x + 4)\,dx$

$= \Big/_{-1}^{2} \left(-\frac{2}{3}x^3 + x^2 + 4x\right) = -\frac{16}{3} + 4 + 8 - \left(\frac{2}{3} + 1 - 4\right)$

$= -\frac{18}{3} + 12 + 3 = 15 - 6 = \underline{9}$

MAA K08

8.



$A = \frac{1}{2} \cdot 4 \cdot 5 \cdot \sin\alpha = 6$

$\sin\alpha = \frac{3}{5}$

$\alpha = 36,869...^\circ \lor \alpha = 180^\circ - 36,8..^\circ \approx 143,1^\circ$

$\alpha = 36,869...^\circ \Rightarrow$

$a^2 = 4^2 + 5^2 - 2 \cdot 4 \cdot 5 \cos\alpha$

$= 16 + 25 - 40\cos\alpha$

$= 9$

$a = \pm 3$

$3^2 + 4^2 = 25 = 5^2 \Rightarrow \gamma = 90^\circ$

tai $\frac{a}{\sin\alpha} = \frac{5}{\sin\gamma} \Leftrightarrow \sin\gamma = \frac{5\sin\alpha}{a} = \frac{5 \cdot \frac{3}{5}}{3} = 1$

$\underline{\underline{V: \text{Suurin kulma on } 90,0^\circ \text{ tai } 143,1^\circ}}$

9 $f(x) = x + \sqrt{9 - x^2}, \quad -3 \le x \le 3$ jatk. derva

$f'(x) = 1 + \frac{-2x}{2\sqrt{9 - x^2}} = \frac{\sqrt{9 - x^2} - x}{\sqrt{9 - x^2}} = 0$

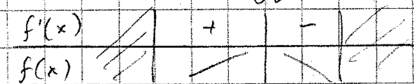
$\sqrt{9 - x^2} = x \quad |()^2, \quad 0 \le x \le 3$

$9 - x^2 = x^2$

$2x^2 = 9$

$x^2 = \frac{9}{2}$

$x = (\pm)\frac{3}{\sqrt{2}}$



$f'(0) = 1$

$f'(2,5) = -0,507..$

Suurin arvo $f\left(\frac{3}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}} - \sqrt{9 - \frac{9}{2}} = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} = \frac{6}{\sqrt{2}} = \frac{6\sqrt{2}}{2}$

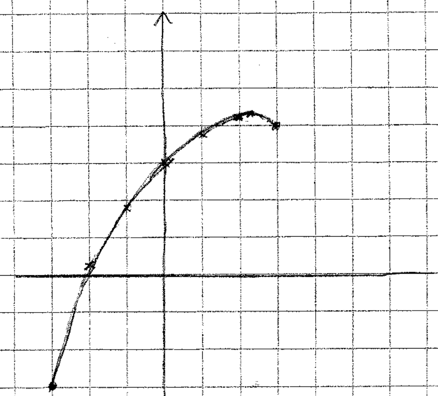
$= 3\sqrt{2} \ (= 4,242...)$

Pienin joko $f(-3) = -3$

tai $f(3) = 3$

$\underline{V: \text{Pienin } f(-3) = -3}$

$\underline{\text{Suurin } f\left(\frac{3}{\sqrt{2}}\right) = 3\sqrt{2}}$



x	y
-3	-3
-2	0,236...
-1	1,82..
0	3
1	3,82...
2	4,23...
3	3

10. $f(x) = e^x - a|x-1|$ jatk. $\forall x \in \mathbb{R}$

$x-1$	$-$	$+$
$\lvert x-1\rvert$	$1-x$	$x-1$
$f(x)$	$e^x - a + ax$	$e^x - ax + a$

$x<1$ $\quad f'(x) = e^x + a \geq 0 \Leftrightarrow a \geq -e^x$

$-e^x$ aid. väh. $\Rightarrow$ suurimmillaan "vasemmalla"

$\lim\limits_{x\to-\infty} -e^x = 0 \Rightarrow$ jos $a \geq 0 \Rightarrow f(x)$ kasv. kun $x<1$

$x>1$ $\quad f'(x) = e^x - a \geq 0 \Leftrightarrow a \leq e^x$

e^x aid. kasv. $\Rightarrow$ pienimmillään "vasemmalla"

$\lim\limits_{x\to1^+} e^x = e \Rightarrow$ jos $a \leq e \Rightarrow f(x)$ kasv. $x \geq 1$

V: $f(x)$ kasvava $\forall x \in \mathbb{R}$, jos $0 \leq a \leq e$

11. a) SYT(154, 126)

$154 : 126 = 1$ jää $28 \Leftrightarrow 154 = 126 + 28 \Leftrightarrow 28 = 154 - 126$

$126 : 28 = 4$ jää $14 \Leftrightarrow 126 = 4\cdot28 + 14 \Leftrightarrow 14 = 126 - 4\cdot28$

$28 : 14 = 2 \Leftrightarrow 28 = 2\cdot14$

$\Rightarrow$ SYT(154, 126) = 14

b) $154x + 126y = 56 = 4\cdot14 \Rightarrow$ yhtälöllä on ratk.

$14 = 126 - 4\cdot28 = 126 - 4\cdot(154-126) = -4\cdot154 + 5\cdot126$

$154\cdot(-4) + 126\cdot5 = 14 \quad |\cdot4$

$154\cdot(-16) + 126\cdot20 = 56$

Yksityisratkaisu $x = -16$, $y = 20$

Yleinen $x = -16 + n\cdot\frac{126}{14} = -16 + 9n \quad n \in \mathbb{Z}$

$y = 20 - n\cdot\frac{154}{14} = 20 - 11n$

TAI $\quad 154 - 126 = 28 \quad |\cdot2$

$154\cdot2 + 126(-2) = 56$

$x = 2 + n\cdot\frac{126}{14} = 2 + 9n$

$y = -2 - n\cdot\frac{154}{14} = -2 - 11n$

V: $(x,y) = (2+9n, -2-11n)$, $n \in \mathbb{Z}$

12. $f(x) = x^2 - 2^x$ jatk.

$f(-1) = 1 - 2^{-1} = \frac{1}{2} > 0$

$f(1) = 1 - 2 = -1 < 0$

f jatk.

$\Rightarrow \exists\, x_0 \in [-1, 1] : f(x_0) = 0$

Haarukoimalla laskimella (taulukko)

$f(-1) = \frac{1}{2} > 0 \qquad f(1) = -1 < 0$

$f(-0{,}7) = -0{,}1255\ldots < 0$

$f(-0{,}77) = 0{,}00648\ldots \qquad f(-0{,}76) = -0{,}128\ldots$

$f(-0{,}767) = 0{,}000650\ldots \qquad f(-0{,}766) = -0{,}00129$

$f(-0{,}76670) = 6{,}85\ldots\cdot10^{-5} \qquad f(-0{,}76660) = -1{,}26\cdot10^{-4}$

$f(-0{,}76667) = 1{,}029\ldots\cdot10^{-5} \qquad f(-0{,}76666) = -9{,}11\cdot10^{-6}$

Näiden välissä

$x \approx -0{,}7667$

Newtonin menetelmä: $\quad f'(x) = 2x - 2^x \ln 2$

$x_0 = -1$

$x_{n+1} = x_n - \dfrac{x_n^2 - 2^{x_n}}{2x_n - 2^{x_n}\ln 2} \qquad (x^n = \text{ANS})$

$x_1 = -0{,}78692\ldots$

$x_2 = -0{,}76684\ldots$

$x_3 = -0{,}76666471008 8 \qquad x \approx -0{,}7667$

$x_4 = -0{,}766664695962$

$x_5 = -0{,}766664695962$

13. $f : \mathbb{R}\to\mathbb{R}$, $g : \mathbb{R}\to\mathbb{R}$ derivoituvia

$f(x) \leq g(x) \Leftrightarrow f(x) - g(x) \leq 0$

a) V: $f'(x) \leq g'(x) \ \forall x \in \mathbb{R}$

Vastaesim.: $f(x) = 2x$, $g(x) = x^2 + 2$

$f(x) - g(x) = 2x - x^2 - 2 = -x^2 + 2x - 2$

ei nollakohtia, koska $D = 4 - 4\cdot(-1)\cdot(-2)$

$= -4 < 0$ ja $\frown$

$\Rightarrow f(x) - g(x) \leq 0 \quad \forall x \in \mathbb{R}$

$f'(x) = 2$

$g'(x) = -2x + 2$

$f'(x) - g'(x) = 2 - (-2x+2) = 2x > 0$, kun $x > 0$

$\Rightarrow$ ei ole $f'(x) \leq g'(x) \ \forall x \in \mathbb{R} \Rightarrow$ Väite epätosi.

b) V: $\int_0^x f(t)\,dt \le \int_0^x g(t)\,dt \quad \forall x \ge 0$

Tod: $\int_0^x f(t)\,dt - \int_0^x g(t)\,dt = \int_0^x [f(t)-g(t)]\,dt \le 0$

koska $f(t)-g(t) \le 0 \quad \forall t \ge 0$

$\Rightarrow \int_0^x f(t)\,dt \le \int_0^x g(t)\,dt \quad \square$

14. $f(x) = \cos x - \sin x$

a) $\cos x - \sin x = 0 \qquad x \in [0, 2\pi]$

$\sin x = \cos x \quad | : \cos x, \; x \ne \frac{\pi}{2} + n\pi$

$\tan x = 1 \Rightarrow \underline{\underline{x = \frac{\pi}{4} \;\vee\; x = \frac{5\pi}{4}}}$

b) $f'(x) = -\sin x - \cos x = 0$

$\Leftrightarrow \tan x = -1 \Rightarrow x = \frac{3\pi}{4} \;\vee\; x = \frac{7\pi}{4}$

$f(x)$ jatk., derivoituva kun $x \in [0, 2\pi]$

suurin ja pienin arvo joko deriv. 0-kohdissa tai välin päissä

$f(0) = 1 - 0 = 1$

$f(\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$

$f(\frac{7\pi}{4}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$f(2\pi) = 1 - 0 = 1$

V: Suurin kun $x = \frac{7\pi}{4}$ ja pienin kun $x = \frac{3\pi}{4}$

c) $\int_0^{2\pi} (\cos x - \sin x)\,dx = \Big/_0^{2\pi} (\sin x + \cos x) = 0 + 1 - (0 + 1) = 0$

d)

	0 ... $\frac{\pi}{4}$	$\frac{\pi}{4}$ ... $\frac{5\pi}{4}$	$\frac{5\pi}{4}$ ... 2π
$\cos x - \sin x$	+	−	+
$\lvert \cos x - \sin x \rvert$	$\cos x - \sin x$	$\sin x - \cos x$	$\cos x - \sin x$

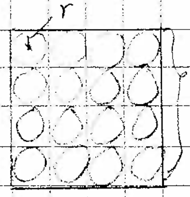
$\int_0^{2\pi} |\cos x - \sin x|\,dx = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx + \int_{\frac{5\pi}{4}}^{2\pi} (\cos x - \sin x)\,dx =$

$\Big/_0^{\frac{\pi}{4}} (\sin x + \cos x) + \Big/_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (-\cos x - \sin x) + \Big/_{\frac{5\pi}{4}}^{2\pi} (\sin x + \cos x)$

$= (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}) - (0 + 1) + (-(-\frac{1}{\sqrt{2}}) - (-\frac{1}{\sqrt{2}})) - (-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}) + (0 + 1) - (-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}})$

$= \frac{8}{\sqrt{2}} = \frac{8\sqrt{2}}{2} = \underline{\underline{4\sqrt{2}}} \approx 5{,}66$

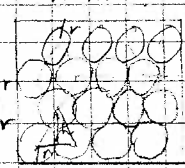
15.



$\frac{A_p}{A_n} = \frac{n^2 \cdot \pi r^2}{(n \cdot 2r)^2} = \frac{\pi n^2 r^2}{4 n^2 r^2} = \underline{\underline{\frac{\pi}{4}}}$

Sama jokaisella n

$n \cdot 2r + r = (2n+1)r$



$(n-1)r\sqrt{3} + 2r$

$= (n\sqrt{3} - \sqrt{3} + 2)r$

$h_\Delta = \frac{2r\sqrt{3}}{2} = r\sqrt{3}$

$A_\ell = (2n+1)(n\sqrt{3} - \sqrt{3} + 2)r^2$

$= r^2(2n^2\sqrt{3} - 2n\sqrt{3} + 4n + n\sqrt{3} - \sqrt{3} + 2)$

$= (2n^2\sqrt{3} + (4-\sqrt{3})n + 2 - \sqrt{3})r^2$

$\frac{A_p}{A_\ell} = \frac{n^2 \pi r^2}{(2n^2\sqrt{3} + (4-\sqrt{3})n + 2 - \sqrt{3})r^2}$

$= \frac{\pi n^2}{2n^2\sqrt{3} + (4-\sqrt{3})n + 2 - \sqrt{3}}$

$n = 10: \quad \frac{A_p}{A_n} = \frac{\pi}{4} = 0{,}7853\ldots \approx \underline{\underline{0{,}79}}$

$\frac{A_p}{A_\ell} = \frac{100\pi}{200\sqrt{3} + 40 - 10\sqrt{3} + 2 - \sqrt{3}} = \frac{100\pi}{189\sqrt{3} + 42} = 0{,}8505\ldots$

$\approx \underline{\underline{0{,}85}}$

b) $\lim_{n\to\infty} \frac{\pi}{4} = \underline{\underline{\frac{\pi}{4} \approx 0{,}79}}$

$\lim_{n\to\infty} \frac{\pi n^2}{2n^2\sqrt{3} + (4+\sqrt{3})n + 2 - \sqrt{3}} = \lim_{n\to\infty} \frac{\pi \cancel{n^2}}{\cancel{n^2}(2\sqrt{3} + \frac{4-\sqrt{3}}{n} + \frac{2-\sqrt{3}}{n^2})}$

$= \underline{\underline{\frac{\pi}{2\sqrt{3}} \approx 0{,}91}}$

Ens. pysyy ennallaan, toinen paranee $\rightarrow$ 91 %