MAA K09

1. a) $\frac{a^2}{3} - \left(\frac{-a}{3}\right)^2 = \frac{3a^2}{9} - \frac{a^2}{9} = \frac{3a^2 - a^2}{9} = \underline{\underline{\frac{2a^2}{9}}}$

b) $(x-3)^2 > (x-1)(x+1)$

$x^2 - 6x + 9 > x^2 - 1$

$-6x > -10 \quad / : (-6)$

$\underline{\underline{x < \frac{5}{3}}}$

c) $\begin{cases} \frac{x}{3} + \frac{y}{2} = 1 \quad / \cdot 6 \\ 3x - 2y + 3 = 0 \end{cases}$

$\begin{cases} 2x + 3y = 6 \quad / \cdot 3 \\ 3x - 2y = -3 \quad / \cdot (-2) \end{cases} \quad / \cdot 2 \quad / \cdot 3$

$\begin{cases} 6x + 9y = 18 \\ -6x + 4y = 6 \end{cases} \qquad \begin{cases} 4x + 6y = 12 \\ 9x - 6y = -9 \end{cases}$

$13y = 24 \qquad 13x = 3$

$y = \frac{24}{13} \qquad x = \frac{3}{13}$

$\underline{\underline{V: \left(\frac{3}{13}, \frac{24}{13}\right)}}$

2. a) $\int_0^1 \sqrt[3]{x}\, dx = \int_0^1 x^{\frac{1}{3}} dx = \left/ \frac{3}{4} x^{\frac{4}{3}} \right|_0^1 = \frac{3}{4} \cdot 1^{\frac{4}{3}} - 0 = \underline{\underline{\frac{3}{4}}}$

b) $(e^x)^3 = e^{x^2}$

$e^{3x} = e^{x^2} \Leftrightarrow 3x = x^2$

$3x - x^2 = 0$

$x(3-x) = 0$

$\underline{\underline{x = 0 \vee x = 3}}$

c) $F(x) = \frac{1}{2}\cos(2x)$

$F'(x) = \frac{1}{2} \cdot 2 \cdot (-\sin 2x) = \underline{\underline{-\sin 2x}}$

3. a) $\vec{a} = 2\vec{i} + 5\vec{j}, \quad \vec{b} = \vec{i} + 2\vec{j}$

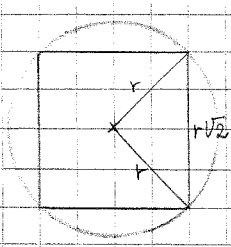
$\vec{a} + \vec{b} = 2\vec{i} + 5\vec{j} + \vec{i} - 2\vec{j} = 3\vec{i} + 3\vec{j}$

$|\vec{a} + \vec{b}| = \sqrt{3^2 + 3^2} = \sqrt{18} = \underline{\underline{3\sqrt{2}}}$

$(\vec{a} + \vec{b})^\circ = \frac{3\vec{i} + 3\vec{j}}{3\sqrt{2}} = \frac{3}{3\sqrt{2}}(\vec{i} + \vec{j}) = \underline{\underline{\frac{1}{\sqrt{2}}(\vec{i} + \vec{j})}}$

MAA K09

3 b)



$\frac{2\pi r}{\frac{1}{2} \pi r \sqrt{2}} = \frac{\pi}{2\sqrt{2}} = 1,1107\ldots \approx 111,1\,\%$

$(111,1 - 100)\,\% = \underline{\underline{11,1\,\%}}$ pitempi

4. $f(x) = 3x^4 - 8x^3 - 18x^2 + 7$

$f'(x) = 12x^3 - 24x^2 + 36x = 0$

$12x(x^2 - 2x - 3) = 0$

$x = 0 \quad \vee \quad x^2 - 2x - 3 = 0$

$x = \frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1} = \frac{2 \pm 4}{2}$

$x = 3 \vee x = -1$

$f(0) = 7 \neq 0$

$f(3) = 3 \cdot 3^4 - 8 \cdot 3^3 - 18 \cdot 3^2 + 7 = -128 \neq 0$

$f(-1) = 3 \cdot (-1)^4 - 8 \cdot (-1)^3 - 18 \cdot (-1)^2 + 7 = 0 \qquad \underline{V: x = -1}$

5. a) $\lg x + \lg(x+30) = 3 \qquad x > 0 \wedge x > -30 \Leftrightarrow \underline{x > 0}$

$\lg x(x+30) = 3$

$\lg(x^2 + 30x) = 3$

$x^2 + 30x = 10^3$

$x^2 + 30x - 1000 = 0$

$x = \frac{-30 \pm \sqrt{900 - 4 \cdot 1 \cdot (-1000)}}{2 \cdot 1} = \frac{-30 \pm 70}{2}$

$x = 20 \quad \vee \quad x = -50$ ei käy $\qquad \underline{V: x = 20}$

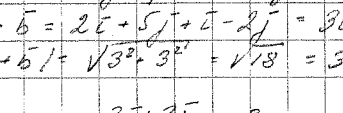
b) $f(x) = \ln(x+1) - \ln x, \quad x > 0$

$f'(x) = \frac{1}{x+1} - \frac{1}{x} = \frac{x - (x+1)}{x(x+1)} = -\frac{1}{x(x+1)} < 0$, kun $x > 0$

$\Rightarrow f(x)$ on aidosti vähenevä, kun $x > 0$

$\Rightarrow f(x)$ on aidosti monotoninen □

6.



a) $P(\text{viallinen}) = P(\text{AV tai BV tai CV})$

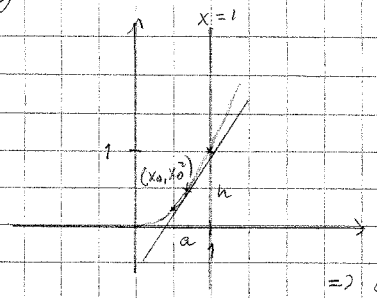
$= 0,6 \cdot 0,02 + 0,3 \cdot 0,03 + 0,1 \cdot 0,04$

$= \underline{\underline{0,025 = 2,5\,\%}}$

b) $P(\text{koneesta C} \mid \text{viallinen})$

$= \frac{0,1 \cdot 0,04}{0,025} = \underline{\underline{0,16 = 16\,\%}}$

7. $y = x^2$, $x_0 \in]0,1]$ $\quad y' = 2x$



$y - x_0^2 = 2x_0(x - x_0)$

$y = 2x_0 x + x_0^2$

$y = 0$

$2x_0 x = x_0^2 \quad | : (2x_0)$

$x = \frac{x_0}{2}$

$\Rightarrow a = 1 - \frac{1}{2}x_0$

$x = 1 \Rightarrow y = 2x_0 - x_0^2$

$A(x) = \frac{1}{2}ah = \frac{1}{2}\left(1 - \frac{x}{2}\right)(2x - x^2)$

$= \frac{1}{2}\left(2x - x^2 - x^2 + \frac{x^3}{2}\right) = \frac{1}{2}\left(2x - 2x^2 + \frac{x^3}{2}\right)$

jatk., eteenpäin $0 < x \le 1$

$A'(x) = \frac{1}{2}\left(2 - 4x + \frac{3}{2}x^2\right) = 0$

$\frac{3}{2}x^2 - 4x + 2 = 0$

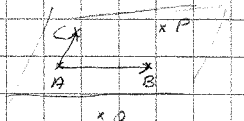
$x = \frac{4 \pm \sqrt{16 - 4 \cdot \frac{3}{2} \cdot 2}}{2 \cdot \frac{3}{2}} = \frac{4 \pm 2}{3}$

$x = 2 \vee x = \frac{2}{3}$

	0		$\frac{2}{3}$		1		2
$A'(x)$		+	0	−		−	
$A(x)$	0	↗	max	↘		↘	

Suurin ala, kun $x_0 = \frac{2}{3}$

8. $A = (3,0,0)$ $B = (0,4,0)$ $C = (1,2,3)$



$\overrightarrow{OP} = \overrightarrow{OA} + s\overrightarrow{AB} + t\overrightarrow{AC}$

$= 3\vec{i} + s\left[(0-3)\vec{i} + (4-0)\vec{j} + (0-0)\vec{k}\right] + t\left[(1-3)\vec{i} + (2-0)\vec{j} + (3-0)\vec{k}\right]$

$= 3\vec{i} + s(-3\vec{i} + 4\vec{j}) + t(-2\vec{i} + 2\vec{j} + 3\vec{k})$

$x\vec{i} + y\vec{j} + z\vec{k} = (3 - 3s - 2t)\vec{i} + (4s + 2t)\vec{j} + 3t\vec{k}$

$\begin{cases} x = 3 - 3s - 2t \\ y = 4s + 2t \\ z = 3t \Rightarrow t = \frac{1}{3}z \end{cases} \Rightarrow 4s = y - \frac{2}{3}z \Leftrightarrow s = \frac{1}{4}y - \frac{1}{6}z$

$x = 3 - \frac{3}{4}y + \frac{1}{2}z - \frac{2}{3}z \quad | \cdot 12$

$12x = 36 - 9y + 6z - 8z$

$12x + 9y + 2z - 36 = 0$

8. TAI $ax + by + cz + d = 0$

$A = (3,0,0)$ $\quad 3a + 0b + 0c + d = 0 \Leftrightarrow 3a + d = 0 \Leftrightarrow a = -\frac{d}{3}$

$B = (0,4,0)$ $\quad 0a + 4b + 0c + d = 0 \Leftrightarrow 4b + d = 0 \Leftrightarrow b = -\frac{d}{4}$

$C = (1,2,3)$ $\quad a + 2b + 3c + d = 0$

$-\frac{d}{3} - 2 \cdot \frac{d}{4} + 3c + d = 0$

$3c = -\frac{d}{6}$

$c = -\frac{d}{18}$

$-\frac{d}{3}x - \frac{d}{4}y - \frac{d}{18}z + d = 0 \quad | \cdot \left(-\frac{36}{d}\right)$

$12x + 9y + 2z - 36 = 0$

9. $\begin{cases} y = \sin\left(\frac{2\pi}{3} - x\right) \\ y = \sin x \end{cases} \quad x \in \mathbb{R}$

$\sin x = \sin\left(\frac{2\pi}{3} - x\right)$

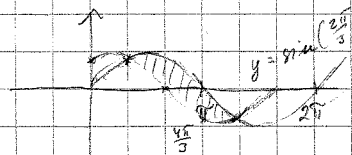
$x = \frac{2\pi}{3} - x + n \cdot 2\pi$ $\quad \vee \quad x = \pi - \left(\frac{2\pi}{3} - x\right) + n \cdot 2\pi$

$2x = \frac{2\pi}{3} + n \cdot 2\pi \quad | : 2$ $\quad x = \frac{\pi}{3} + x + n \cdot 2\pi$ ei ratk.

$x = \frac{\pi}{3} + n \cdot \pi$

$\sin\left(\frac{\pi}{3} + n \cdot 2\pi\right) = \frac{\sqrt{3}}{2}$ $\quad \sin\left(\frac{\pi}{3} + \pi + n \cdot 2\pi\right) = \sin\left(\frac{4\pi}{3} + n \cdot 2\pi\right) = -\frac{\sqrt{3}}{2}$

Yht. pisteet $\left(\frac{\pi}{3} + n \cdot 2\pi, \frac{\sqrt{3}}{2}\right)$ ja $\left(\frac{4\pi}{3} + n \cdot 2\pi, -\frac{\sqrt{3}}{2}\right)$



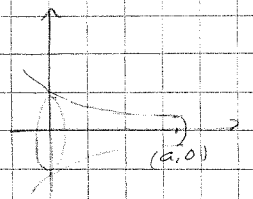
x	$\sin\left(\frac{2\pi}{3} - x\right)$
0	$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$
$\frac{\pi}{3}$	$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$
$\frac{2\pi}{3}$	$\sin 0 = 0$
π	$\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$A = \int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} \left(\sin x - \sin\left(\frac{2\pi}{3} - x\right)\right) dx$

$= \left[-\cos x - \cos\left(\frac{2\pi}{3} - x\right)\right]_{\frac{\pi}{3}}^{\frac{4\pi}{3}} = -\cos\frac{4\pi}{3} - \cos\left(-\frac{2\pi}{3}\right) - \left(-\cos\frac{\pi}{3} - \cos\frac{\pi}{3}\right)$

$= \frac{1}{2} + \frac{1}{2} - \left(-\frac{1}{2} - \frac{1}{2}\right) = 2$

10. $f(x) = e^{-x}$ $x \in [0, a]$



$$V(a) = \pi \int_0^a e^{-2x}\,dx$$

$$= \pi \cdot \left(-\frac{1}{2}\right) \int_0^a -2e^{-2x}\,dx$$

$$= -\frac{\pi}{2} \Big/_0^a e^{-2x} = -\frac{\pi}{2}\left(e^{-2a} - e^0\right) = \underline{\frac{\pi}{2}\left(1 - e^{-2a}\right)}$$

$$V_\infty = \lim_{a \to \infty} \frac{\pi}{2}\left(1 - e^{-2a}\right) = \underline{\frac{\pi}{2}}$$

$$\frac{\pi}{2}\left(1 - e^{-2a}\right) = 0{,}99 \cdot \frac{\pi}{2}$$

$$1 - e^{-2a} = 0{,}99$$

$$e^{-2a} = 0{,}01 \quad / \ln$$

$$-2a = \ln 0{,}01$$

$$a = -\frac{\ln 0{,}01}{2} = 2{,}302\ldots \approx \underline{2{,}3}$$

11. $\dfrac{9n^2 + 117n + 34}{3n + 5} \in \mathbb{Z}_+$, $n \in \mathbb{Z}_+$

$$\begin{array}{r|l} & 3n + 34 \\ \hline 3n + 5 & 9n^2 + 117n + 34 \\ -9n^2 \mp 15n & \\ \hline & 102n + 34 \\ & \mp 102n \mp 170 \\ \hline & -136 \end{array}$$

$$\frac{9n^2 + 117n + 34}{3n + 5} = 3n + 34 - \frac{136}{3n + 5} \in \mathbb{Z}, \text{ jos}$$

$\dfrac{136}{3n+5} \in \mathbb{Z}$ $\quad 3n + 5 = 136$

$3n = 131 \Rightarrow n \notin \mathbb{Z}$

$136 = 2 \cdot 68 = 4 \cdot 34 = 8 \cdot 17$

$3n + 5 = 2 \Leftrightarrow 3n = -3 \Leftrightarrow n = -1 \in \mathbb{Z}_+$

$3n + 5 = 4 \Leftrightarrow 3n = -1 \Rightarrow n \notin \mathbb{Z}$

$3n + 5 = 34 \Leftrightarrow 3n = 29 \Rightarrow n \notin \mathbb{Z}$

$3n + 5 = 68 \Leftrightarrow 3n = 63 \Leftrightarrow n = 21 \in \mathbb{Z}_+$

$3n + 5 = 8 \Leftrightarrow 3n = 3 \Leftrightarrow n = 1 \in \mathbb{Z}_+$

$3n + 5 = 17 \Leftrightarrow 3n = 12 \Leftrightarrow n = 4 \in \mathbb{Z}_+$

$\underline{V: 1, 4, 21}$

MAA K09

12. $x^3 = x + 2 \Leftrightarrow x^3 - x - 2 = 0$

$f(x) = x^3 - x - 2$ jatk., derivoi

$f'(x) = 3x^2 - 1 = 0$

$x^2 = \frac{1}{3}$

$x = \pm \frac{1}{\sqrt{3}}$

	$-\frac{1}{\sqrt{3}}$		$\frac{1}{\sqrt{3}}$	
$f'(x)$	+	−	+	
$f(x)$	/	\	/	

$f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{3\sqrt{3}} - \frac{1}{3} - 2 < 0$

$f\left(-\frac{1}{\sqrt{3}}\right) = -\frac{1}{3\sqrt{3}} + \frac{1}{3} - 2 < 0 \Rightarrow$ funktiolla on

vain 1 juuri (kun $x > \frac{1}{\sqrt{3}}$)

$x_0 = 1$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - x_n - 2}{3x_n^2 - 1} = \frac{3x_n^3 - x_n - x_n^3 + x_n + 2}{3x_n^2 - 1}$$

$$= \frac{2x_n^3 + 2}{3x_n^2 - 1}$$

$x_1 = 2$

$x_2 = 1{,}636\ldots$

$x_3 = 1{,}530392\ldots$

$x_4 = 1{,}521444\ldots$

$x_5 = 1{,}521379\,21$

$x_6 = 1{,}521379\,07$

$x \approx \underline{1{,}52}$

Koska $f(x)$ on jatkuva,

$f(1) = 1 - 1 - 2 = -2 < 0$

$f(2) = 2^3 - 2 - 2 = 4 > 0$

$f'(x) > 0$ kun $x \geq 1$

$\Rightarrow$ funktiolla on täsm. yksi nollakohta välillä $[1, \infty[$

$\Rightarrow$ yhtälöllä on täsm. 1 juuri $\in [1, \infty[$ □

13. Osasumma $S_n = \sum_{i=1}^{n} a_i$. Sarja suppenee, jos $\exists S \in \mathbb{R}$:

$\lim_{n \to \infty} S_n = S$

V: $\sum_{n=1}^{\infty} S_n$ ei suppene, jos $a_i > 0$ $\forall i \in \mathbb{Z}_+$

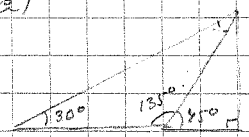
Tod. $a_i > 0$ $\forall i \in \mathbb{Z}_+ \Rightarrow S_{n+1} - S_n = a_{n+1} > 0$

$\Rightarrow$ jono S_n on aidosti kasv. ja $S_n \geq a_1 > 0$

$\Rightarrow$ ei voi olla $\lim_{n \to \infty} S_n = 0 \Rightarrow \sum_{n=1}^{\infty} S_n$ ei suppene □

MAA
K09

14. a)



$$\frac{h}{a+h} = \tan 30° = \frac{1}{\sqrt{3}}$$

$$h\sqrt{3} = a + h$$

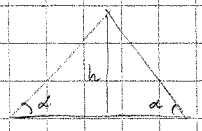
$$h(\sqrt{3}-1) = a$$

$$h = \frac{\sqrt{3}+1}{\sqrt{3}-1}a = \frac{\sqrt{3}+1}{2}a$$

b) $V = \frac{1}{3}\cdot a^2 \cdot \frac{\sqrt{3}+1}{2}\cdot a$

$$= \frac{\sqrt{3}+1}{6}a^3$$

c)



$$\tan\alpha = \frac{h}{\frac{1}{2}a} = \frac{\frac{1}{2}(\sqrt{3}+1)a}{\frac{1}{2}a} = \sqrt{3}+1$$

$$\alpha = 69,89..° \approx \underline{70°}$$

15. $[0,1]$ f derivoituva, $f'(x) \geq 2$ $\forall x \in [0,1]$, $\int_0^1 f(x)\,dx = 1$

a) väliarvolause $\Rightarrow \exists x_0 \in]0,1[$:

$$f(x) - f(0) = f'(x_0)(x-0) \quad \forall x \in [0,1]$$

$$\Rightarrow f(x) - f(0) \geq 2x \Leftrightarrow f(x) \geq f(0) + 2x \quad \square$$

b) $$1 = \int_0^1 f(x)\,dx \geq \int_0^1 (f(0)+2x)\,dx = \Big[f(0)\,x + x^2\Big]_0^1$$

$$= f(0) + 1 - 0 = f(0) + 1$$

$$1 \geq f(0) + 1 \Leftrightarrow f(0) \leq 0 \quad \square$$

c) $f(0) \leq 0$, $f'(x) \geq 2 > 0 \Rightarrow$ f aidosti kasvava $\Rightarrow$ korkeintaan 1 nollakohta

$\int_0^1 f(x)\,dx = 1 > 0 \Rightarrow \exists x_0 \in]0,1[: f(x_0) > 0$

f jatkuva, $f(0) \leq 0$ ja $f(x_0) > 0 \Rightarrow$ ainakin 1 nollakohta

$x_1 \in [0, x_0[\subset [0,1]$ $\therefore$ f:llä tasan. 1 nollakohta $[0,1]$