MAA – K2011 Ratkaisut

1. a) $\frac{2}{x} = \frac{3}{x-2}$ Määrittely $x \neq 0$ ja $x-2 \neq 0$, $x \neq 2$

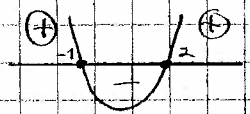
$2(x-2) = 3x$
$2x - 4 = 3x$
$\underline{\underline{x = -4}}$

b) $x^2 - 2 \leq x$
$x^2 - x - 2 \leq 0$

Nollakohdat:
$x^2 - x - 2 = 0$

$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-2)}}{2 \cdot 1}$

$= \frac{1 \pm 3}{2} = \begin{cases} -1 \\ 2 \end{cases}$



Vast: $-1 \leq x \leq 2$

c) $\left|\frac{3}{2}x - 6\right| = 6$

$\frac{3}{2}x - 6 = 6$ tai $\frac{3}{2}x - 6 = -6$

$\frac{3}{2}x = 12 \quad || \cdot \frac{2}{3}$ $\frac{3}{2}x = 0$

$\underline{\underline{x = 8}}$ tai $\underline{\underline{x = 0}}$

2. a) $35{,}50\,€ \cdot 1{,}12 \cdot 0{,}9 = 35{,}784$ (€)

Arvon nousu $\frac{35{,}784 - 35{,}50}{35{,}50} = 0{,}008$ V: Arvo nousi 0,8%

b) $k = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-1}{5-(-2)} = \frac{-4}{7} = \underline{\underline{-\frac{4}{7}}}$

c) $e^{5\ln 2 - \ln 8} = e^{\ln 2^5 - \ln 8}$
$= e^{\ln 32 - \ln 8}$
$= e^{\ln \frac{32}{8}}$
$= e^{\ln 4}$
$= \underline{\underline{4}}$

3. $f(x) = xe^{-x^2}$ $g(x) = 2e^{-x^2}$

a) $f(x) = g(x)$

$xe^{-x^2} = 2e^{-x^2} \quad || : e^{-x^2} \ (\neq 0!)$
$\underline{\underline{x = 2}}$

b) $f'(x) = 1 \cdot e^{-x^2} + x \cdot e^{-x^2} \cdot (-2x)$
$= e^{-x^2} - 2x^2 e^{-x^2} \Rightarrow f'(1) = e^{-1^2} - 2 \cdot 1^2 \cdot e^{-1^2}$
$= e^{-1} - 2e^{-1} = \underline{\underline{-\frac{1}{e}}}$

c) $\int_0^1 f(x)\,dx = \int_0^1 xe^{-x^2}\,dx$
$= -\frac{1}{2}\int_0^1 -2xe^{-x^2}\,dx$
$= -\frac{1}{2} \Big/_0^1 e^{-x^2}$
$= -\frac{1}{2}\left(e^{-1^2} - e^{-0^2}\right)$
$= -\frac{1}{2}\left(\frac{1}{e} - 1\right)$
$= \underline{\underline{-\frac{1}{2e} + \frac{1}{2}}} \quad (\approx 0{,}316)$

4. $f(x) = 2^x$. Olkoon kysytty polynomi $g(x) = ax^2 + bx + c$. $(a \neq 0)$

$\begin{cases} g(0) = f(0) \\ g(1) = f(1) \\ g(2) = f(2) \end{cases}$ $\begin{cases} a \cdot 0^2 + b \cdot 0 + c = 2^0 = 1 & \Rightarrow c = 1 \\ a \cdot 1^2 + b \cdot 1 + 1 = 2^1 = 2 & \Rightarrow a + b = 1 \\ a \cdot 2^2 + b \cdot 2 + 1 = 2^2 = 4 & \Rightarrow 4a + 2b = 3 \end{cases}$

$\begin{cases} a + b = 1 \quad || \cdot (-2) \\ 4a + 2b = 3 \end{cases}$

$+\begin{cases} -2a - 2b = -2 \\ 4a + 2b = 3 \end{cases}$
$2a = 1$

$\underline{a = \frac{1}{2}}$

$a + b = 1$
$b = 1 - \frac{1}{2} = \frac{1}{2}$

$g(x) = \frac{1}{2}x^2 + \frac{1}{2}x + 1$

Vast: Polynomi on
$\underline{\underline{\frac{1}{2}x^2 + \frac{1}{2}x + 1}}$

5.

Merkitään $f(x) = x(x+3)(5-x)$
$= (x^2+3x)(5-x)$
$= 5x^2 - x^3 + 15x - 3x^2$
$= -x^3 + 2x^2 + 15x$

$f'(x) = -3x^2 + 4x + 15$

$f'(x) = 0 \iff -3x^2 + 4x + 15 = 0$

$$x = \frac{-4 \pm \sqrt{4^2 - 4\cdot(-3)\cdot 15}}{2\cdot(-3)} = \frac{-4 \pm 14}{-6} = \begin{cases} -\frac{5}{3} \notin [-1,5] \\ 3 \end{cases}$$

Kulkukaavio välillä $[-1,5]$

	$-\frac{5}{3}$	-1	3	5
$f'(x)$			+	−
$f(x)$			↗	↘

Suurin arvo on $f(3) = 3(3+3)(5-3) = \underline{\underline{36}}$

$f(-1) = (-1)(-1+3)(5-(-1)) = -12$ P A.

$f(5) = 5(5+3)(5-5) = 0$ V: Pienin arvo on -12, suurin arvo on 36.

6.

$$P(0 \text{ oikein}) = \frac{\binom{7}{3}}{\binom{10}{3}} = \frac{35}{120} = \underline{\underline{\frac{7}{24}}} \quad (\approx 0{,}2917)$$

← 3 vääristä

↖ 3 arvotaan 10:stä

1 oikea

10:stä

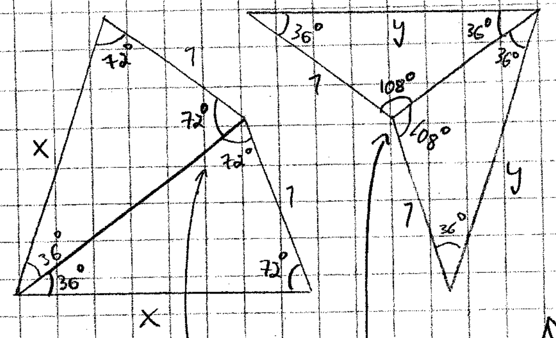
$$P(1 \text{ oikein}) = \frac{\binom{3}{1}\binom{7}{2}}{\binom{10}{3}} = \frac{3\cdot 21}{120} = \underline{\underline{\frac{21}{40}}} \quad (= 0{,}525)$$

$$P(2 \text{ oikein}) = \frac{\binom{3}{2}\binom{7}{1}}{\binom{10}{3}} = \frac{3\cdot 7}{120} = \underline{\underline{\frac{7}{40}}} \quad (= 0{,}175)$$

$$P(3 \text{ oikein}) = \frac{\binom{3}{3}}{\binom{10}{3}} = \underline{\underline{\frac{1}{120}}} \quad (\approx 0{,}0083)$$

Todennäköisyyksien summa $= \frac{7}{24} + \frac{21}{40} + \frac{7}{40} + \frac{1}{120} = \underline{1}$

7.



Puolitetaan kulma
⇒ Kolmiot ovat yhtenevät
⇒ Molemmissa puuttuva sivu = x

(sivu = y)

a)

$$\frac{x}{\sin 72^\circ} = \frac{1}{\sin 36^\circ}$$

$\sin 36^\circ\, x = \sin 72^\circ$

$$x = \frac{\sin 72^\circ}{\sin 36^\circ} = 1{,}61803\ldots$$

$\underline{\underline{x \approx 1{,}618}}$

$$\frac{y}{\sin 108^\circ} = \frac{1}{\sin 36^\circ}$$

$$y = \frac{\sin 108^\circ}{\sin 36^\circ} = 1{,}61803\ldots$$

$\underline{\underline{y \approx 1{,}618}}$

b) Kolmion alan trigonometrisella kaavalla

$A_{leija} = 2\cdot\frac{1}{2}\cdot 1\cdot x\cdot \sin 72^\circ$
$= 1{,}61803\ldots \cdot \sin 72^\circ$
$= 1{,}53884\ldots$
$\approx \underline{\underline{1{,}539}}$

$A_{Nuoli} = 2\cdot\frac{1}{2}\cdot 1\cdot y\cdot \sin 36^\circ$
$= 1{,}61803\ldots \cdot \sin 36^\circ$
$= 0{,}9[illegible]5\ldots \approx \underline{\underline{0{,}951}}$

8.

$\bar{a} = 4\bar{i} - 5\bar{j} + 3\bar{k}$ ja $\bar{b} = 2\bar{i} + \bar{j} - 2\bar{k}$.

Halutaan $\bar{a} = \bar{u} + \bar{v}$. $\Rightarrow \bar{v} = \bar{a} - \bar{u}$

$\bar{u} = t\bar{b}$

$$\bar{v}\cdot\bar{b} = 0$$
$$(\bar{a} - \bar{u})\cdot\bar{b} = 0$$
$$(\bar{a} - t\bar{b})\cdot\bar{b} = 0$$
$$\bar{a}\cdot\bar{b} - t\bar{b}\cdot\bar{b} = 0$$

$$4\cdot 2 + (-5)\cdot 1 + 3\cdot(-2) - t\left(2\cdot 2 + 1\cdot 1 + (-2)(-2)\right) = 0$$
$$-3 - 9t = 0$$
$$t = -\frac{1}{3}$$

$$\Rightarrow \bar{u} = t\bar{b} = -\frac{1}{3}(2\bar{i} + \bar{j} - 2\bar{k}) = -\frac{2}{3}\bar{i} - \frac{1}{3}\bar{j} + \frac{2}{3}\bar{k}$$

$$\bar{v} = \bar{a} - \bar{u} = 4\bar{i} - 5\bar{j} + 3\bar{k} - \left(-\frac{2}{3}\bar{i} - \frac{1}{3}\bar{j} + \frac{2}{3}\bar{k}\right) = \frac{14}{3}\bar{i} - \frac{14}{3}\bar{j} + \frac{7}{3}\bar{k}$$

$$\Rightarrow \bar{a} = \left(-\frac{2}{3}\bar{i} - \frac{1}{3}\bar{j} + \frac{2}{3}\bar{k}\right) + \left(\frac{14}{3}\bar{i} - \frac{14}{3}\bar{j} + \frac{7}{3}\bar{k}\right)$$

9.

$a_1 = \frac{5}{4}$, $a_n = -\frac{3}{4}a_{n-1}$, $n = 2, 3, \ldots$

a) Jono on geometrinen, jonka suhdeluku $q = -\frac{3}{4}$.

$$\Rightarrow a_n = a_1 \cdot q^{n-1} \Leftrightarrow a_n = \frac{5}{4}\cdot\left(-\frac{3}{4}\right)^{n-1}$$

b) $\sum_{n=1}^{\infty} a_n$ on suppeneva geometrinen sarja, sillä

$$|q| = \left|-\frac{3}{4}\right| = \frac{3}{4} < 1.$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n = \frac{a_1}{1-q} = \frac{\frac{5}{4}}{1-(-\frac{3}{4})} = \frac{5}{4}\cdot\frac{4}{7} = \frac{5}{7}$$

10.

$f: [0, 2\pi] \to \mathbb{R}$ jva.

Koska ei voida tietää kumpi on arvoltaan suurempi $f(x)$ vai $f(x) + \sin x$, niin integroidaan erotuksen itseisarvo!

$$A = \int_0^{2\pi} |f(x) + \sin x - f(x)|\,dx = \int_0^{2\pi} |\sin x|\,dx$$

$$|\sin x| = \begin{cases} \sin x, & \text{kun } 0 \le x \le \pi \\ -\sin x, & \text{kun } \pi < x \le 2\pi \end{cases} \Rightarrow$$

$$= \int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} -\sin x\,dx$$

$$= \Big/_0^{\pi} -\cos x + \Big/_{\pi}^{2\pi} \cos x\,dx$$

$$= -\cos\pi - (-\cos 0) + \cos 2\pi - \cos\pi$$

$$= -(-1) - (-1) + 1 - (-1)$$

$$= 4$$

11.

$$f(x) = \begin{cases} ax^2, & x \le -1 \\ \frac{x^2}{1+x^2}, & x > -1 \end{cases}$$

a) $f(x)$ on jatkuva kaikkialla, jos $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(-1)$.

(Muualla $f(x)$ on selvästi jatkuva ts. $\lim_{x\to x_0} f(x) = f(x_0)$, kun $x_0 < -1$ tai $x_0 > -1$.)

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} ax^2 = a(-1)^2 = a = f(-1)$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{x^2}{1+x^2} = \frac{1}{1+1} = \frac{1}{2}$$

$\Rightarrow$ oltava $a = \frac{1}{2}$.

Jatkuu 11.

b) Selvästi $f(x)$ on derivoituva, kun $x \neq -1$.

$\Rightarrow$ Riittää tarkastella kohdan $x = -1$ derivoituvuutta.

Toispuoleiset derivaatat oltava samat

$$\Rightarrow \lim_{x \to -1^-} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1^-} \frac{\frac{1}{2}x^2 - \frac{1}{2}}{x+1} = \lim_{x \to -1^-} \frac{\frac{1}{2}(x^2 - 1^2)}{x+1}$$

$$= \lim_{x \to -1^-} \frac{\frac{1}{2}(x-1)\cancel{(x+1)}}{\cancel{x+1}} = \underline{\underline{-1}}$$

$$\lim_{x \to -1^+} \frac{\frac{x^2}{1+x^2} - \frac{(-1)^2}{1+(-1)^2}}{x-(-1)} = \lim_{x \to -1^+} \frac{\frac{x^2}{1+x^2} - \frac{1}{2}}{x+1}$$

$$= \lim_{x \to -1^+} \frac{\frac{2x^2 - 1 - x^2}{2(1+x^2)}}{x+1}$$

$$= \lim_{x \to -1^+} \frac{x^2 - 1}{2(1+x^2)} \cdot \frac{1}{x+1}$$

$$= \lim_{x \to -1^+} \frac{(x-1)\cancel{(x+1)}}{2(1+x^2)\cancel{(x+1)}} = \frac{-2}{4} = \underline{\underline{-\frac{1}{2}}}$$

Toispuoleiset derivaatat ovat erisuuret $\left(-1 \neq -\frac{1}{2}\right)$,

joten $f(x)$ ei ole derivoituva kohdassa $x = -1$.

V: $f(x)$ ei ole derivoituva kaikkialla.

11. Jatkuu

c) $$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2}{1+x^2}$$

$$= \lim_{x \to \infty} \frac{\cancel{x^2}}{\cancel{x^2}\left(\frac{1}{x^2}+1\right)} = \frac{1}{1} = \underline{\underline{1}}$$

$\frac{1}{x^2} \to 0$

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12. $46^{78} + 89^{67}$ jaollinen viidellä?

$46 \equiv 1 \pmod 5$ ja $89 \equiv -1 \pmod 5$

$$\left(\begin{array}{l} 46 = 9 \cdot 5 + 1 \\ 1 = 0 \cdot 5 + 1 \end{array}\right) \text{ kongruentit mod 5} \qquad \left(\begin{array}{l} 89 = 17 \cdot 5 + 4 \\ -1 = -1 \cdot 5 + 4 \end{array}\right) \text{ kongruentit mod 5}$$

$$\Rightarrow 46^{78} + 89^{67} \equiv 1^{78} + (-1)^{67} \pmod 5$$

$$\equiv 1 - 1 \pmod 5$$

$$\equiv 0 \pmod 5$$

$\Rightarrow$ Jaettaessa 5:llä jakojäännös on nolla!

$\Rightarrow$ Vastaus: luku on jaollinen 5:llä.

13. $P(x) = 2x^4 - x^3 + x^2 - x - 1$

Huomataan (laskin?), että $x = -\frac{1}{2}$ ja $x = 1$ ovat $P(x)$:n nollakohtia. Osoitetaan tämä:

$$P(-\tfrac{1}{2}) = 2\cdot(-\tfrac{1}{2})^4 - (-\tfrac{1}{2})^3 + (-\tfrac{1}{2})^2 - (-\tfrac{1}{2}) - 1$$
$$= \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{4} + \tfrac{1}{2} - 1$$
$$= \underline{\underline{0}}$$

$$P(1) = 2\cdot 1^4 - 1^3 + 1^2 - 1 - 1 = \underline{0}$$

$\Rightarrow$ Eräs tekijä on $(x+\frac{1}{2})(x-1) = x^2 - \frac{1}{2}x - \frac{1}{2}$

$$\begin{array}{r|l}
 & 2x^2 \quad\; + 2 \\
\hline
x^2 - \frac{1}{2}x - \frac{1}{2} & 2x^4 - x^3 + x^2 - x - 1 \\
 & -(2x^4 - x^3 - x^2) \\
\hline
 & \qquad\quad 2x^2 - x - 1 \\
 & \qquad\quad 2x^2 - x - 1 \\
\hline
 & \qquad\qquad\qquad 0
\end{array}$$

Merkitään $Q(x) = 2x^2 + 2$

$$2x^2 + 2 = 0$$
$$x^2 = -1 \quad \text{ei ratk.}$$

$\Rightarrow Q(x)$:llä ei ole nollakohtia

$\Rightarrow Q(x)$ ei jakaannu enää tekijöihin

$\Rightarrow$ Vast. $2x^4 - x^3 + x^2 - x - 1 = (x+\frac{1}{2})(x-1)(2x^2+2)$

$$= (x+\tfrac{1}{2})(x-1)2\cdot(x^2+1)$$
$$= \underline{\underline{(x^2+1)(2x+1)(x-1)}}$$

14.* $f(x) =$ [redacted]x, $g(x) = 1 - \frac{x^2}{2}$.

a) Väite: $f(x) \geq g(x)$ kaikilla x.

Tod. Tutkitaan erotusfunktiota $h(x) = f(x) - g(x)$.

$$h(x) = \cos x - (1 - \tfrac{x^2}{2}) = \cos x + \tfrac{1}{2}x^2 - 1$$

$$h'(x) = -\sin x + x$$

$$h'(x) = 0$$
$$-\sin x + x = 0$$
$$\sin x = x$$

Eräs nollakohta on $x = 0$ ($\sin 0 = 0$).

Osoitetaan, että $h'(x)$:llä ei ole muita nollakohtia.

$$h''(x) = \underbrace{-\cos x + 1}_{\in [-1+1,\, 1+1] = [0,2]}$$

sillä $-1 \leq \cos x \leq 1$

Näin ollen $h''(x) \geq 0$ ja $h''(x) = 0$ yksittäisissä kohdissa $x = n\cdot 2\pi$.

$\Rightarrow h'(x)$ on aidosti kasvava

$\Rightarrow h'(x)$:llä ei voi olla muita nollakohtia kuin $x = 0$.

		0	
$h'(x)$	$-$		$+$
$h(x)$	$\searrow$		$\nearrow$

$h'(-1) = -\sin(-1) + (-1) \approx -0{,}159 < 0$

$h'(1) = -\sin 1 + 1 \approx 0{,}159 > 0$

$\Rightarrow h(x)$:n pienin arvo on

$$h(0) = \cos 0 + \tfrac{1}{2}\cdot 0^2 - 1$$
$$= 1 + 0 - 1$$
$$= 0$$

Näin ollen $h(x) \geq 0$

$\Rightarrow f(x) - g(x) \geq 0$

$\Rightarrow f(x) \geq g(x)$ $\square$

14. Jatkuu...

b) $f(x) = g(x) \iff h(x) = 0.$

a) -kohdan perusteella $h(x) = 0$, kun $x = 0$.
Muilla x:n arvoilla $h(x) \neq 0$ (kulkukaavio!)

$\Rightarrow$ Vast: $x = 0.$

c) $h(x) = f(x) - g(x)$. Kulkukaavion perusteella suurin arvo on joko $h(-\pi)$ tai $h(\pi)$.

$$h(-\pi) = \cos(\pi) + \tfrac{1}{2}(\pi)^2 - 1 = \tfrac{1}{2}\pi^2 - 2$$

$$h(\pi) = \cos\pi + \tfrac{1}{2}\pi^2 - 1 = \tfrac{1}{2}\pi^2 - 2$$

$\Rightarrow$ Erotuksen suurin arvo on $\underline{\underline{\tfrac{1}{2}\pi^2 - 2}}$. $(\approx 2{,}935)$

d) $$A = \int_{-\pi}^{\pi} f(x) - g(x)\,dx = \int_{-\pi}^{\pi} \cos x + \tfrac{1}{2}x^2 - 1\,dx$$

a)-kohta $f(x) \geq g(x)$

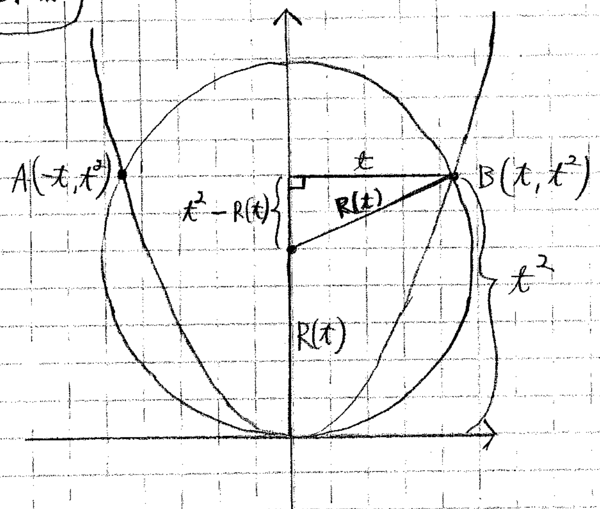
$$= \Big/_{-\pi}^{\pi} \sin x + \tfrac{1}{6}x^3 - x$$

$$= \sin\pi + \tfrac{1}{6}\pi^3 - \pi - \left(\sin(-\pi) + \tfrac{1}{6}(-\pi)^3 - (-\pi)\right)$$

$$= 0 + \tfrac{1}{6}\pi^3 - \pi - 0 + \tfrac{1}{6}\pi^3 - \pi$$

$$= \underline{\underline{\tfrac{1}{3}\pi^3 - 2\pi}} \quad (\approx 4{,}05)$$

15. *



a) Kuvan suorakulmaisesta kolmiosta

$$(t^2 - R(t))^2 + t^2 = R(t)^2$$

$$t^4 - 2t^2R(t) + \cancel{R(t)^2} + t^2 = \cancel{R(t)^2}$$

$$-2t^2R(t) = -t^4 - t^2 \quad \| : (-2t^2)$$

$$R(t) = \underline{\underline{\tfrac{1}{2}t^2 + \tfrac{1}{2}}}$$

b) $$R_0 = \lim_{t\to 0^+} R(t) = \lim_{t\to 0^+}\left(\tfrac{1}{2}t^2 + \tfrac{1}{2}\right) = 0 + \tfrac{1}{2} = \underline{\underline{\tfrac{1}{2}}}$$

c) Rajaympyrän $KP = (0, R_0) = (0, \tfrac{1}{2})$ ja säde $R_0 = \tfrac{1}{2}$.

$\Rightarrow$ yhtälö $(x - 0)^2 + (y - \tfrac{1}{2})^2 = (\tfrac{1}{2})^2$

$$x^2 + y^2 - y + \tfrac{1}{4} = \tfrac{1}{4}$$

$$y^2 - y + x^2 = 0$$

$$\iff y = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\cdot 1\cdot x^2}}{2\cdot 1} = \frac{1 \pm \sqrt{1 - 4x^2}}{2}$$

Alapuoli saadaan miinusmerkillä

$$\Rightarrow \underline{\underline{g(x) = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - 4x^2}}}$$

75 jatkuu

d) $g(x) = \frac{1}{2} - \frac{1}{2}\sqrt{1-4x^2} = \frac{1}{2} - \sqrt{\frac{1}{4} - x^2}$

$g'(x) = -\cancel{\frac{1}{2}}\left(\frac{1}{4} - x^2\right)^{-\frac{1}{2}} \cdot (-\cancel{2}x)$

$= \frac{x}{\sqrt{\frac{1}{4} - x^2}}$

$g''(x) = \frac{1 \cdot \sqrt{\frac{1}{4} - x^2} - x\sqrt{\frac{1}{4} - x^2}}{\left(\frac{1}{4} - x^2\right)}$

$g''(0) = \frac{\frac{1}{2} - 0}{\frac{1}{4}} = \underline{\underline{2}}$

$f(x) = x^2 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = \underline{\underline{2}}$

$\Rightarrow f''(0) = \underline{\underline{2}}$

$\frac{1}{R_0} = \frac{1}{\frac{1}{2}} = \underline{\underline{2}}$

$\Rightarrow g''(0) = f''(0) = \frac{1}{R_0} = \underline{\underline{2}}$ □